THE ESTABLISHED RESULTS OF THE NEW PSYCHOLOGY AS IT BEARS UPON THE TEACHING OF MATHEMATICS.

By Agnes L. Rogers.

By the New Psychology we indicate the comparatively modern psychology based upon experiment and measurement. Its results have been obtained from investigations, which have been conducted under controlled conditions and which can therefore be repeated by other investigators and verified. The older analytical psychology, which used the method of introspection, made valuable contributions to our knowledge of the structure of mental processes; but from a practical point of view it is more important to know their dynamical character and functional relations and this is particularly true as regards teaching, since the fundamental problem for education is "the production and prevention of changes in human beings."

Psychologists have made, however, relatively few experimental investigations in the field of mathematics. If we disregard arithmetic, which has of late received considerable attention from the experimentalists, we find that most of the publications upon the psychology of the subject have used the older methods of introspection and observation. This is not surprising, when we recall how little is known as yet of the psychology of thought and how indispensable systematic observation is as a preliminary to experimentation. The result is that we have at present various theories of a speculative character as to the nature of mathematical ability, which still await confirmation. Möbius, for example, has advanced the view that mathematical talent is a special capacity, independent of other mental capacities and characterized by unusual ability in understanding relations of number, in judging relations of size and in concrete imagery. Hüther, on the other hand, maintains that mathematical genius involves no specific, fundamental capacity; it consists merely in an exceptional ease in carrying out certain thought operations. It involves marked development of con-

crete imagery, synthetic imagination, and mathematical understanding. Betz, who agrees with Hüther's general theory, offers a different explanation of mathematical ability. He contends that the mathematical type of mind is characterized by a special clearness of certain "minimal" or highly abstract ideas, and by the ability to manipulate and vary these with ease and precision. Again, Henri Poincaré, the distinguished mathematician, on the basis of introspective analysis gives it as his judgment that mathematical ability has nothing to do with a very sure memory or a special power of attention. It is a feeling for order and the concealed relations of numbers that distinguishes the mathematicians from other men. He divides mathematical reasoners, however, into two distinct classes, the geometrical or intuitional and the analytical or logical types.

All such theories are interesting, but singularly barren of fruitful practical suggestions. At best they merely indicate to the experimental psychologists promising subjects of research. It should not be inferred, however, that experimentation alone can furnish information of value to the mathematics teacher. On the the contrary, such a method of attack as that of Judd furnishes an illuminating account of the mental activities that mathematics demands. It presents an excellent survey of the simpler psychological processes underlying mathematics, which have been experimentally investigated, describing typical mental reactions involved in mathematical thinking and analyzing the psychological implications of the text-books in use and of current class-room procedure. In comparison, the New Psychology has little to offer; but by its refinement of method it undoubtedly gives promise of richer results.

Early experimental effort was naturally directed to analysis of the mental processes involved in the simplest branches of the subject, namely arithmetical operations. The work in this field has been both extensive and significant for mathematics in general. For our purpose the most important results of the statistical studies by Rice, Thorndike, Stone, Bonser, Courtis, Winch, and Starch* are the demonstration of the wide range of individual differences in capacity and the evidence in support of the specialization and independence of the different abilities involved

* See Bibliography, H. B. Howell, "A Foundational Study in the Pedagogy of Arithmetic."

in arithmetic. The extent of individual differences had already been shown by investigations upon other mental functions; its demonstration in the case of arithmetic, however, was exceptionally striking. Equally remarkable was the discovery that a high degree of excellence in the fundamental processes may be present along with a low degree of skill in arithmetical reasoning and vice versa. Indeed, it was found that a similar variability prevailed among the fundamental processes themselves. These results led Fox and Thorndike to prophesy that the abilities tested—addition, multiplication, fractions, rational computation and problems—bear little resemblance to those of the mathematician.

The amount of experimental investigation accomplished in algebra and geometry falls far short of the work done in arithmetic. If we exclude the recent efforts to establish standards for algebra by Thorndike, Monroe, and Rugg, a new line of activity, which is bound to have considerable effect upon the teaching of algebra, we find that in all the experimental investigations published, with two exceptions, the data have been school and college marks or class lists. Correlations have been calculated between mathematical ability as a whole and ability in other school subjects† and in brief the result has been that fairly high positive correlations have been obtained.

An interesting attempt to secure a more complete and detailed analysis of mathematical intelligence was made in 1910 by William Brown. He used the same statistical method of correlations, obtaining his data from a school examination in algebra, geometry and arithmetic. He corrected the papers, however, by two methods, first according to ordinary school standards, and second according to a differential system of marking, based on an introspective analysis of the intellectual processes involved in answering. His principal results are, first, that geometrical ability and algebraic ability are not related, save through their connection with arithmetical ability, which is of some interest with reference to the present effort to correlate these subjects more closely, and second, that memory of preceding propositions is the central ability in geometry, being related most intimately to other forms of geometrical ability. This is in harmony with

† See Columbia Contributions to Philosophy, Vol. XI., No. 2.

his opinion that school mathematics and higher mathematics relate to different forms of ability and it raises the whole question of the distinction between school mathematics as it *is*, and as it *might be*. The fact that it is now advocated in some quarters that the concepts of higher mathematics should be introduced into the secondary school suggests that even if Brown's conclusion is true of the present state of affairs, it need not be with different methods of teaching and other standards. In any event his experimental procedure should lead us to accept his statements with caution; for obviously the "psychologizing" of examination papers is an unsatisfactory manner of measurement, and further the number of individuals examined was relatively small.

Another statistical study carried out in the Dartmouth pedagogical department under the direction of F. C. Lewis, deserves mention on account of the new departure in method. Instead of using ordinary school marks as data, tests were given in originals in geometry and in practical reasoning and the scores made in these were correlated. It may be doubted whether these tests were adequate measures of the abilities in question; but the mode of procedure certainly marks a step in advance and the results are noteworthy. The pupils of each of twenty-four groups were arranged in two series, the first according to their ranking in mathematical reasoning and the second according to their ranking in practical reasoning. It was found that of the first five mathematical reasoners from each group, 63 per cent. were at the foot of the practical reasoning series, conspicuous for their inefficiency in practical reasoning, and of the pupils at the foot of the mathematical reasoning series, 47 per cent. were conspicuous for their positions at the head of the practical reasoning series.

These statistical inquiries like the earlier accounts based on introspection and observation are interesting rather than helpful. Their general outcome can be briefly expressed in the statement that *a high correlation exists between efficiency in mathematics and efficiency in other subjects.*

For further guidance we must turn to general psychology and here the crucial question is transfer of training and for two reasons. Not only do experiments upon transfer yield the most

useful suggestions for methods of teaching; upon them ultimately rests the defence of the place accorded to mathematics in the curriculum of the secondary school. For the mathematicians themselves have admitted that all the facts that a skilled mechanic or engineer would ever need could be taught in a few lessons. Consequently very little of the mathematics at present given in high school could be retained on the ground of its practical usefulness. Nor does the conventional value of mathematics justify the time and effort it entails. Society does not regard an individual as grossly ignorant or ill-informed, if his knowledge of mathematics is extremely meager. Current opinion rather assumes that mathematical skill is highly specialized and unrelated to general intelligence or culture. The ultimate defence of the retention of mathematics in the curriculum rests therefore upon its general educational value.

What then are the established results as regards transfer of training? *At the present time no psychologist of repute denies that there is transfer.* Experimentation has conclusively shown that practice in one function affects other functions. The points at issue are the extent to which transfer takes place and the methods by which it is secured. *It has been established that the amount of transfer varies with the degree of difference between the functions in question.* Change either in the content or in the method of study reduces the extent of the spread of improvement. Accordingly the indirect effect of practice is invariably less than the direct. Furthermore transfer can be negative: the habits or mental acts developed by a particular kind of training may inhibit rather than facilitate other mental activities. Investigators have found very different degrees of transfer effect in accordance with the different functions tested and the different experimental conditions and frequently it has been extremely small in amount; but on the whole there is ground for the statement that if transfer is often not largely general, it is probably always to some extent general and "a very small spread of training may be of great educational value, provided it covers a wide enough field." As Thorndike points out, "*if a hundred hours of training in being scientific about chemistry produces only one hundredth as much improvement in being scientific about all sorts of facts,*

it would be a very remunerative educational force." Certain facts have also to be borne in mind in reference to the smallness of the transfer obtained in experimental investigations. For the most part the latter have used adults as subjects and in their case we would not expect to obtain as much transfer as in the case of children. Whereas in the young and immature mental habits are still in process of formation, in adults these are practically established and in consequence any improvement made in training is probably due to adaptation to special conditions and therefore not susceptible of generalization. The results of Dallenbach's experiments on visual apprehension in school children contrast strongly with those of Whipple and Foster on adults as regards practice effects and cast doubt upon the common assumption that conclusions derived from a limited number of selected adults are necessarily true of growing children. Again, certain features of the experiments made on transfer seem distinctly unfavorable to spread of improvement. Thus the practice periods have usually been short and the training given can be fairly described as work at high pressure. On general educational grounds we do not believe that improvements so effected are likely to transfer. Further, the processes tested in the laboratory investigations are comparatively simple. They differ in marked fashion from the complex processes involved in Latin, mathematics or science. In short, the conditions of the experiments depart so radically from ordinary class-room conditions, it may well be questioned whether results so obtained can possibly determine even approximately the amount of transfer possible in the case of the school subjects. Even where experiments have been carried out in the schoool room, as by Winch and Sleight, the methods of securing greatest transfer have not been fully utilized. It seems reasonable, therefore, to conclude that even if but slight transfer effect has been found in experimental work, yet judgment against a wide spread of improvement, in general, should be suspended, since the conditions favorable to generalization were absent. At present we have only prophecy, not knowledge. The experimental results to hand are so paltry and limited compared with the mass of facts to be measured in any reliable attempt to establish the extent of transfer in the case of any of the secondary school subjects, that

an extreme view stands discredited. It is credible, to say the least, that the amount of general effect produced by the high-school training in mathematics is greatly in excess of the laboratory figures. But measurement of the actual changes made by mathematics has still to be accomplished.

Psychology has a more positive contribution to make as regards the second point in dispute. There is, to be sure, a variety of opinions as to the ways and means by which transfer is facilitated. Identical elements, development of attention, will, mental imagery, ideals, divesting the essential process of inessential elements, improvement in the technique of learning—all of these have been suggested as causes of transfer and probably all do function to some extent in the spread of improvement. There is, however, a growing consensus of opinion upon the factors operative in transfer, as a result of the progress made by those investigators, who have subjected their numerical and introspective results to careful analysis. Prominent among such studies is that of Ruger. In this article his conclusions can only be briefly summarized. He found four general factors in transfer of training—ideals, attitudes, concepts of method and high level attention. The formation of ideals, such as a general idea of efficiency, was an important element in the spread of improvement from one activity to another, and similarly the attitude adopted facilitated or hindered transfer. Thus a self-conscious attitude restricted progress and checked transfer, while an attitude of self-confidence was very favorable. Again, concepts of method played a most important part. The conscious control of assumptions, the active search for new hypotheses, the effort to distinguish between suggestions and to classify them appropriately, the deliberate testing of hypotheses, and the generalization of these methods themselves together with the realization of the value of such generalizations greatly contributed to transfer. Above all a high level of attention was an indispensable precondition. Of the special factors in transfer the most important were related ideas. Upon these depends to a large extent the possibility of generalization. We have been apt to believe that if we gave individuals a theory, they would be able to apply it appropriately; but the fact is that children have to be as carefully taught to apply theory as to un-

derstand it and unless related ideas are pointed out to them they will often fail to perceive their connection. Ruger found in his experiments with wire puzzles that geometrical concepts played no part in hastening their solution and that the greatest transfer was from similar puzzles. Finally as regards the relation of habit to transfer it seemed that "the mere presence in the case of change of conditions of motor habits appropriate to the new conditions did not necessitate positive transfer. The degree of positive transfer varied directly with the precision of analysis of the similarity of the new case to the old."

These results suggest the following practical injunctions. First, proper attitudes should be cultivated in the pupil. Secondly attention should be focused on the art of learning and on methods of procedure in the solving of problems, so that they should be stimulated to analyze the situation, to formulate hypotheses, to criticize and evaluate each suggestion, to be systematic in selecting and rejecting these and in verifying them. Further, each step should be generalized as a method so that there should be deliberate control of assumptions. Thirdly, attention should be directed to related ideas in order that as many as possible may be recalled or discovered. Lastly, motivation should be secured and attention be kept at a high level. By such means and in such ways the experience gained in mathematics will tend to be generalized and made available in other fields.

BIBLIOGRAPHY.

Möbius, P. J. Über die Anlage zur Mathematik, Leipzig, 1900.

Hüther, A. Über das Problem einer psychologischen und pädagogischen Theorie der intellektuellen Begabung. *Archiv für die gesamte Psychologie,* 1910, p. 193.

Betz, W. Über Korrelationen. *Zeitschrift für Angewandte Psychologie,* Beihefte, 1911.

Poincaré, H. L'Invention Mathématique. *Bulletin de l'Institut général psychologique,* 1908.

—— Sur la Nature du Raisonnement Mathématique. *Revue de Métaphysique et de Morale,* II., 371–384.

Judd, C. H. Psychology of High School Subjects, 1915. Chapters II.–VI.

Howell, H. B. A Foundational Study in the Pedagogy of Arithmetic, 1914.

Monroe, W. S. A Test of the Attainment of First-Year High-School Students in Algebra. *The School Review,* March, 1915, XXIII., No. 3.

RUGG, H. O. The Experimental Determination of Standards in First Year Algebra. *School Review,* January, 1916, Vol. XXIV., No. 1.

RIETZ AND SHADE. Correlation of Efficiency in Mathematics and Efficiency in Other Subjects. The University of Illinois Studies, Vol. 6, No. 10, 1908.

THORNDIKE, SMITH AND BURRIS. Heredity, Correlation and Sex Differences in School Abilities. Columbia Contributions to Philosophy, Vol. XI., No. 2.

BROWN, WM. An Objective Study of Mathematical Intelligence. *Biometrika,* 1910, No. 7, 352–367.

LEWIS, F. C. A Study in Formal Discipline. *School Review,* April, 1905.

DALLENBACH, K. M. The Effect of Practice upon Visual Apprehension in School Children. *Journal of Educational Psychology,* 1914.

WHIPPLE, G. M. The Effect of Practice upon the Range of Visual Attention and of Visual Apprehension. *Journal of Educational Psychology,* pp. 249–262, 1910.

FOSTER, W. S. The Effect of Practice upon Visualizing and upon the Reproduction of Visual Impressions. *Journal of Educational Psychology,* 1911, 11–21.

RUGER, H. A. The Psychology of Efficiency. *Archives of Psychology,* No. 15, 1910.

TEACHERS COLLEGE,
NEW YORK CITY.